

Massive phonon modes from a BEC-based analog model

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Two-component BECs subject to laser-induced coupling exhibit a complicated spectrum of excitations, which can be viewed as two interacting phonon modes. We study the conditions required to make these two phonon modes decouple. Once decoupled, the phonons not only can be arranged to travel at different speeds, but one of the modes can be given a mass — it exhibits the dispersion relation of a massive relativistic particle $\omega = \sqrt{\omega_0^2 + c^2 k^2}$. This is a new and unexpected excitation mode for the coupled BEC system. Apart from its intrinsic interest to the BEC community, this observation is also of interest for the “analogue gravity” programme, as it opens the possibility for using BECs to simulate massive relativistic particles in an effective “acoustic geometry”.

cond-mat/0409639

24 September 2004; L^AT_EX-ed February 2, 2008

Analogue models for gravitation can be used to simulate classical and quantum field theory in curved space-time. The first analogue model for black holes, and for simulating Hawking evaporation was suggested by Bill Unruh [1]. He demonstrated that a sound wave propagating through a converging fluid flow exhibits the same kinematics as light does in the presence of a curved space-time background. Since then several other media have been analyzed, and the field has developed tremendously. The first approach specifically using Bose–Einstein condensates as an analogue model was made nineteen years later [2]. Since then various configurations of BECs have been studied to simulate different scenarios for gravity [3, 4, 5, 6, 7, 8]. Until now it has only been possible to simulate light, (generally speaking, massless relativistic particles), propagating through a curved space-time [9, 10, 11, 12, 13]. In the following a two-species BEC is used to extend the class of equations that can be simulated to the full curved-space Klein–Gordon equations. In the language of the BEC community — we have developed a way of giving a mass to the phonon.

The system we will use in our theoretical analysis is an

ultra cold 2-component BEC atomic gas. For example, a two-component condensate of ^{87}Rb atoms in different hyperfine levels, which we label $|A\rangle$ and $|B\rangle$. (Experiments using two different spin states, $|F=1, m=-1\rangle$ and $|F=2, m=2\rangle$, were first performed at *JILA* in 1999 [14].) At zero temperature nearly all atoms occupy the ground state. Then the quantized field describing the microscopic system can be replaced by a classical mean-field, a macroscopic wave-function. In this so-called mean-field approximation the number of non-condensed atoms is small. Interactions between the condensed and non-condensed atoms are neglected in the mathematical description, but two-particle collisions between condensed atoms are included. In the case of a two-component system, interactions within each species (U_{AA} , U_{BB}) and between the different species ($U_{AB} = U_{BA}$) take place. In addition the two condensates are coupled by a laser-beam, which drives transitions between the two hyperfine states with a constant rate λ . (Without the coupling λ no mass term is generated, which is consistent with [8].) The resulting coupled time-dependent Gross–Pitaevskii equations are:

$$\begin{aligned} i \hbar \partial_t \Psi_A &= \left[-\frac{\hbar^2}{2m_A} \nabla^2 + V_A^* + U_{AA} |\Psi_A|^2 + U_{AB} |\Psi_B|^2 \right] \Psi_A + \lambda \Psi_B, \\ i \hbar \partial_t \Psi_B &= \left[-\frac{\hbar^2}{2m_B} \nabla^2 + V_B^* + U_{BB} |\Psi_B|^2 + U_{AB} |\Psi_A|^2 \right] \Psi_B + \lambda \Psi_A, \end{aligned} \quad (1)$$

where $V_{A/B}^* = V_{A/B} - \mu_{A/B}$ denotes the combined effects of the external potential $V_{A/B}$ and the chemical potential $\mu_{A/B}$ [15, 16]. In the eikonal approximation the two stationary background states are described by their den-

sities $\{\rho_A, \rho_B\}$ and phases $\{\theta_A, \theta_B\}$:

$$\Psi_X = \sqrt{\rho_X} e^{i(\theta_X)} \quad \text{for } X = A, B. \quad (2)$$

These four variables are in general not independent of time and space.

In the following, we study zero sound in the overlap region of the two-component system, produced by exciting density perturbations which are small compared to the density of each condensate cloud. In the first experiment studying localized excitations in a one-component Bose–Einstein condensate [17], a laser beam was used to generate a small modulation in the density. Using a phase-contrast imaging it was shown that the resulting perturbation corresponds to a sound wave. The observed speed of sound is

$$c(r) = \sqrt{\frac{4\pi\hbar^2 a \rho(r)}{m^2}} = \sqrt{\frac{U \rho(r)}{m}}, \quad (3)$$

where $\rho(r)$ is the density of the ground state, a is the scattering length, m is the atomic mass and U is the self-interaction constant. The mathematical equations describing these perturbation leads to the well-known hydrodynamic equations, which are the basis for the most fruitful of the analogies between condensed matter physics and general relativity [1, 2, 9, 12].

The same method can be used to obtain the kinematic equations for small perturbations propagating in a two-component system. Given that the density modulation is small, the perturbations in the densities and phases can be linearized around their unperturbed macroscopic states $\{\rho_{A0}, \rho_{B0}\}$:

$$\Psi_X = \sqrt{\rho_{X0} + \varepsilon \rho_{X1}} e^{i(\theta_{X0} + \varepsilon \theta_{X1})} \quad \text{for } X = A, B. \quad (4)$$

These states still satisfy the coupled Gross–Pitaevskii equation. After a straightforward calculation the terms of first order in ε representing the sound waves include two coupled equations for the perturbation of the phases

$$\begin{aligned} \dot{\theta}_{A1} &= -\vec{v}_{A0} \nabla \theta_{A1} - \frac{\tilde{U}_{AA}}{\hbar} \rho_{A1} + \frac{\tilde{U}_{AB}}{\hbar} \rho_{B1}, \\ \dot{\theta}_{B1} &= -\vec{v}_{B0} \nabla \theta_{B1} - \frac{\tilde{U}_{BB}}{\hbar} \rho_{B1} + \frac{\tilde{U}_{AB}}{\hbar} \rho_{A1}. \end{aligned} \quad (5)$$

Here $\vec{v}_{A0} = (\hbar/m_A) \nabla \theta_{A0}$ and $\vec{v}_{B0} = (\hbar/m_B) \nabla \theta_{B0}$ are the background velocities for the macroscopic ground states, while

$$\begin{aligned} \tilde{U}_{AA} &= U_{AA} - \frac{\lambda}{2} \frac{\sqrt{\rho_{B0}}}{\sqrt{\rho_{A0}}^3}, \\ \tilde{U}_{BB} &= U_{BB} - \frac{\lambda}{2} \frac{\sqrt{\rho_{A0}}}{\sqrt{\rho_{B0}}^3}, \\ \tilde{U}_{AB} &= U_{AB} - \frac{\lambda}{2} \frac{1}{\sqrt{\rho_{A0} \rho_{B0}}}, \end{aligned} \quad (6)$$

are modified interaction potentials for the two coupled condensates. In addition to these two phase equations, there are two coupled equations for the density perturbations

$$\begin{aligned} \dot{\rho}_{A1} &= -\nabla \cdot \left(\frac{\hbar}{m_A} \rho_{A0} \nabla \theta_{A1} + \rho_{A1} \vec{v}_{A0} \right) + \frac{2\lambda}{\hbar} \sqrt{\rho_{A0} \rho_{B0}} (\theta_{B1} - \theta_{A1}), \\ \dot{\rho}_{B1} &= -\nabla \cdot \left(\frac{\hbar}{m_B} \rho_{B0} \nabla \theta_{B1} + \rho_{B1} \vec{v}_{B0} \right) + \frac{2\lambda}{\hbar} \sqrt{\rho_{A0} \rho_{B0}} (\theta_{A1} - \theta_{B1}). \end{aligned} \quad (7)$$

It is useful to define the coupling matrix

$$\Xi = \frac{1}{\hbar} \begin{pmatrix} \tilde{U}_{AA} & -\tilde{U}_{AB} \\ -\tilde{U}_{AB} & \tilde{U}_{BB} \end{pmatrix}. \quad (8)$$

A second coupling matrix can be introduced as

$$\Lambda = \frac{2\sqrt{\rho_{A0} \rho_{B0}}}{\hbar} \begin{pmatrix} +\lambda & -\lambda \\ -\lambda & +\lambda \end{pmatrix}, \quad (9)$$

which vanishes completely if the coupling laser is switched off. Last but not least, it is also useful to introduce the background velocity matrix V (a 2×2 matrix of 3-vectors)

$$V = \begin{pmatrix} \vec{v}_{A0} & 0 \\ 0 & \vec{v}_{B0} \end{pmatrix}, \quad (10)$$

and the mass-density matrix D

$$D = \hbar \begin{pmatrix} \frac{\rho_{A0}}{m_A} & 0 \\ 0 & \frac{\rho_{B0}}{m_B} \end{pmatrix}. \quad (11)$$

Collecting terms into a 2×2 matrix equation, the equations for the phases (5) and densities (7) become

$$\dot{\bar{\theta}} = -\Xi \cdot \bar{\rho} - V \cdot \nabla \bar{\theta}, \quad (12)$$

$$\dot{\bar{\rho}} = -\nabla \cdot (D \cdot \nabla \bar{\theta} + V \cdot \bar{\rho}) - \Lambda \cdot \bar{\theta}, \quad (13)$$

where $\bar{\theta}^T = (\theta_{A1}, \theta_{B1})$ and $\bar{\rho}^T = (\rho_{A1}, \rho_{B1})$.

Equation (12) can be used to eliminate $\bar{\rho}$ and $\dot{\bar{\rho}}$ in equation (13), leaving us with a single matrix equation

for the perturbed phases:

$$\partial_t \left(\Xi^{-1} \cdot \dot{\bar{\theta}} \right) = -\partial_t \left(\Xi^{-1} \cdot V \cdot \nabla \bar{\theta} \right) - \nabla \left(V \cdot \Xi^{-1} \cdot \dot{\bar{\theta}} \right) + \nabla \left[(D - V \cdot \Xi^{-1} \cdot V) \nabla \bar{\theta} \right] + \Lambda \cdot \bar{\theta} \quad (14)$$

This equation tells us how a localized collective excitation in a two-component system with a permanent coupling — if λ is nonzero — develops in time. So far there are no restrictions on the masses (m_A, m_B), densities (ρ_{A0}, ρ_{B0}), background velocities ($\vec{v}_{A0}, \vec{v}_{B0}$), interaction constants (U_{AA}, U_{BB}, U_{AB}), and coupling constant (λ). In the eikonal approximation this differential equation leads to the Fresnel equation

$$\det \left\{ \omega^2 \Xi^{-1} - \omega \left[\Xi^{-1} \cdot (V\mathbf{k}) + (V\mathbf{k}) \cdot \Xi^{-1} \right] - [D k^2 - (V\mathbf{k}) \cdot \Xi^{-1} \cdot (V\mathbf{k})] + \Lambda \right\} = 0, \quad (15)$$

which is in general a quartic dispersion relation for two interacting phonon modes. Here $V\mathbf{k}$ denotes the matrix

$$V\mathbf{k} = \begin{pmatrix} \vec{v}_{A0} \cdot \vec{k} & 0 \\ 0 & \vec{v}_{B0} \cdot \vec{k} \end{pmatrix}. \quad (16)$$

The first step in analyzing equation (14) is to ask whether it is possible to decouple the system into two independent phonon modes. We have found decoupling is not possible without introducing several constraints on the background quantities.

Focusing on the last term in equation (14), the eigenvectors for a non-zero coupling $\lambda \neq 0$ are given by $\{[1, 1], [-1, 1]\}$ and the corresponding eigenvalues are $\{0, 4\lambda\sqrt{\rho_{A0}\rho_{B0}}/\hbar\}$. The eigenvectors are fixed and independent of any of the other physical variables. As a result the only way to decouple equation (14) into two phonon modes, is to decompose it in the following way:

$$\bar{\theta} = \begin{pmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_1 \end{pmatrix} + \begin{pmatrix} -\tilde{\theta}_2 \\ \tilde{\theta}_2 \end{pmatrix} \quad (17)$$

We now analyze equation (14) term by term with respect to this decomposition.

The term on the LHS has the same eigenvectors as equation (17) if and only if $\tilde{U}_{AA} = \tilde{U}_{BB}$. The eigenvalues of Ξ^{-1} corresponding to $\{[1, 1], [-1, 1]\}$ are $\{\hbar/(\tilde{U}_{AA} - \tilde{U}_{AB}), \hbar/(\tilde{U}_{AA} + \tilde{U}_{AB})\}$. This places another constraint on the interaction variables: $\tilde{U}_{AA} \neq \tilde{U}_{AB}$. (Further discussion regarding this tight requirement will be presented later.)

The eigenvectors for the first two terms on the RHS in equation (14) can be found in one step, by using the fact that they have simultaneous eigenvectors if and only if the commutator $[\Xi^{-1}, V] = 0$ vanishes. Therefore $\vec{v}_{A0} = \vec{v}_{B0} = \vec{v}_0$, the background velocities must be equal if the two phonon modes are to decouple. The corresponding eigenvalues are those from the matrix Ξ^{-1} multiplied by \vec{v}_0 . In other words, the backgrounds of two condensates must be in phase $\theta_{A0} = \theta_{B0}$.

We are now left with the penultimate term in equation (14). Because $V = \vec{v}_0 \cdot \mathbf{1}$, and the eigenvectors of Ξ^{-1} are already known, there is only the mass-density matrix D to consider. The last constraint to decouple the equation for the two phases then is $\hbar\rho_{A0}/m_A = \hbar\rho_{B0}/m_B = d$. So $D = d \cdot \mathbf{1}$ has for both eigenvectors the eigenvalue $d = \hbar\sqrt{\rho_{A0}\rho_{B0}}/\sqrt{m_A m_B}$.

Applying all this to equation (14) one obtains two decoupled equations for the phonon modes described by the eigenstates (17):

$$\partial_\mu (f_I^{\mu\nu} \partial_\nu \tilde{\theta}_I) = \frac{4\lambda\sqrt{\rho_{A0}\rho_{B0}}}{\hbar} \delta_{2I} \tilde{\theta}_I, \quad \text{for } I = 1, 2. \quad (18)$$

Here

$$f_I^{\mu\nu} = \frac{d}{c_I^2} \left(\begin{array}{c|c} -1 & -v_0^j \\ \hline -v_0^j & c_I^2 \delta^{ij} - v_0^i v_0^j \end{array} \right), \quad (19)$$

where the propagation speeds are defined in terms of the eigenvalues Ξ_I of the matrix Ξ by

$$c_I^2 = \Xi_I d = \frac{d(\tilde{U}_{AA} + (-1)^I \tilde{U}_{AB})}{\hbar}. \quad (20)$$

Introducing effective “spacetime metrics” by the identifications $\sqrt{-g_I} g_I^{\mu\nu} = f_I^{\mu\nu}$ and $g_I = 1/\det[g_I^{\mu\nu}]$, we can recast these equations as a pair of curved-space Klein-Gordon [massive d’Alembertian] equations

$$\frac{1}{\sqrt{-g_I}} \partial_\mu \left(\sqrt{-g_I} g_I^{\mu\nu} \partial_\nu \tilde{\theta}_I \right) - \mathbf{m}^2 \delta_{2I} \tilde{\theta}_I = 0. \quad (21)$$

Here

$$g_I^{\mu\nu} = \left(\frac{d}{c_I} \right)^{-2/(D-2)} \left[\frac{1}{c_I^2} \left(\begin{array}{c|c} -1 & -v_0^j \\ \hline -v_0^j & c_I^2 \delta^{ij} - v_0^i v_0^j \end{array} \right) \right], \quad (22)$$

which depends on the space-time dimension D [9, 13] in such a manner that

$$g_{\mu\nu}^I = \left(\frac{d}{c_I} \right)^{2/(D-2)} \left(\begin{array}{c|c} -(c_I^2 - v_0^2) & -v_0^j \\ \hline -v_0^j & \delta^{ij} \end{array} \right). \quad (23)$$

Finally the mass-term is

$$\mathbf{m}^2 = -\frac{4\lambda\sqrt{\rho_{A0}\rho_{B0}}}{\hbar} (c_2^2/d^D)^{1/(D-2)}, \quad (24)$$

corresponding to the natural oscillation frequency

$$\begin{aligned} \omega_0^2 &= \mathbf{m}^2 c_2^2 \left(\frac{d}{c_2} \right)^{2/(D-2)} = -\frac{4\lambda\sqrt{\rho_{A0}\rho_{B0}} c_2^2}{\hbar d} \\ &= -\frac{4\lambda\sqrt{m_A m_B} c_2^2}{\hbar^2}. \end{aligned} \quad (25)$$

The dispersion relation (in the eikonal limit) is then

$$(\omega - \mathbf{v}_0 \cdot \mathbf{k})^2 - c_I^2 k^2 = \omega_0^2 \delta_{2I}. \quad (26)$$

(A similar calculation, but restricted to a one-condensate system, where all variables are allowed to be time and space dependent, but no mass term is present, has been presented in [6].) Equation (21) is the curved-space Klein-Gordon equation (massive d'Alembertian equation). For $\hat{\theta}_1$, (corresponding to perturbations in the two condensates A and B oscillating “in phase”), the mass term is always zero. However, for a laser-coupled system ($\lambda \neq 0$) the mass-term in the equation for $\hat{\theta}_2$, (corresponding to perturbations in the two condensates A and B oscillating “in anti-phase”), does not vanish.

Comparing the definition for the speed of sound (3) in a one component system, with the c_I introduced here, we see that the c_I (20) are the modified speeds of sound for each phonon mode. (If $U_{AB} = \lambda = 0$ the two condensates decouple and the c_I limit to the independent phonon speeds in each condensate cloud.) This fact leaves us with the possibility of constructing two different types of analog model. So far we have been dealing with a two-metric structure, which is interesting in itself [11, 18]. For instance, in the absence of laser-coupling ($\lambda = 0$) the presence of two different speed of sounds can be used for tuning effects [8].

If we wish to more accurately simulate the curved spacetime of our own universe, another constraint should be placed on the system, to make the two speeds of sound equal $c_1 = c_2$. This yields a single sound-cone structure, to match the observed fact that our universe exhibits a single light-cone structure. This condition is fulfilled if we set $\bar{U}_{AB} = 0$. While the in-phase perturbations will propagate exactly at the speed of sound,

$$\mathbf{v}_s = \mathbf{v}_0 + \hat{\mathbf{k}} c_I, \quad (27)$$

the anti-phase perturbations will move with a lower group velocity given by:

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \mathbf{v}_0 + \hat{\mathbf{k}} \frac{c_I^2}{\sqrt{\omega_0^2 \delta_{2I} + c_I^2 k^2}}. \quad (28)$$

Here \mathbf{k} is the usual wave number. This explicitly demonstrates that the group velocity of the anti-phase eigenstate depends on the laser-induced coupling between the condensates.

In conclusion, the calculation presented in this article is of interest to two separate communities. For the BEC community, it provides a specific example of how to tune an interacting 2-BEC condensate in such a manner as to obtain a massive phonon. Without the fine tuning, it provides an example of two interacting phonon modes whose dispersion relation is governed by the quartic Fresnel equation (15). For the general relativity community, this article provides an example of an analogue system that can be used to mimic a minimally coupled scalar field embedded in a curved spacetime.

Acknowledgements: This research was supported by the Marsden Fund administered by the Royal Society

of New Zealand. We wish to thank Crispin Gardiner, Piyush Jain, and Ashton Bradley for their thoughtful comments.

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